

# Spherical Gravitational Collapse: Tangential Pressure and Related Equations of State

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## Abstract

We derive an equation for the acceleration of a fluid element in the spherical gravitational collapse of a bounded compact object made up of an imperfect fluid. We show that non-singular as well as singular solutions arise in the collapse of a fluid initially at rest and having only a tangential pressure. We obtain an exact solution of Einstein equations, in the form of an infinite series, for collapse under tangential pressure with a linear equation of state. We show that if a singularity forms in the tangential pressure model, the conditions for the singularity to be naked are exactly the same as in the model of dust collapse.

# 1 Introduction

The study of gravitational collapse of a bounded spherical object in classical general relativity has received major attention over the last few years. The purpose of these investigations has been twofold - to establish whether or not naked singularities arise in gravitational collapse [1], and to study the occurrence of critical phenomena in gravitational collapse [2].

Various models of spherical collapse have been studied over the last few years, and these show that both black holes and naked singularities arise during gravitational collapse. The collapsing matter is assumed to satisfy one or more energy conditions, and is in this sense regarded as physically reasonable matter. The models studied so far include collapse of dust [3], null dust [4], perfect [5] and imperfect fluids [6], and scalar fields [7]. In each of these cases, the formation of covered as well as naked singularities has been observed. There are also demonstrations that both these types of solutions will arise in the collapse of a general form of matter [8]. Also, a naturalness argument has recently been put forth to suggest that covered as well as naked singular solutions arise generically in spherical collapse, subject to the assumption of the dominant energy condition [9].

Critical behaviour has been discovered in the collapse of matter fields and fluids, largely through numerical studies. In some of these studies, it has been observed that the solution separating dispersive solutions from collapsing ones is a naked singularity. It should however be noted that in studies of critical behaviour, the numerical identification of a “black hole” is carried out by the detection of an apparent horizon. The possibility that some of these “black hole” solutions are actually naked singularities cannot be a priori ruled out. If the singularity is globally naked, there would be a Cauchy horizon lying outside the apparent horizon, in the Penrose diagram. Such a singularity cannot be identified in present numerical

studies, which do not probe the central high curvature region of the collapsing object. On the other hand, a few analytical studies of scalar collapse confirm that the non-dispersive solutions contain both black holes and naked singular solutions [10].

In the present paper, we study the spherical gravitational collapse of an imperfect fluid under the assumption that the radial pressure is identically zero, but the tangential pressure is non-zero. This system has been studied by a few authors in the past [11], and also more recently [12], with the recent studies focusing on the issue of naked singularity formation. The works mentioned in [12] give evidence for naked singularity formation. The purpose of the present paper is to demonstrate the occurrence of non-singular and covered and naked singular solutions in a specific tangential pressure model. As we will show, our analytical results are similar, in some respects, to numerical results of critical behaviour in collapse. We consider an equation of state for the tangential pressure  $p_T$ , of the form  $p_T = k(r)\rho$ , for which we obtain an exact solution of the Einstein equations, in the form of an infinite series. For physical reasons, we assume that the function  $k(r)$  must vanish at the origin of coordinates. As a consequence of this constraint we find that if a singularity forms in this model (which is not always the case) the conditions for the occurrence of a naked singularity are exactly the same as in the dust collapse model.

The focus of our investigation will be the singularity that may possibly form during the collapse, at the origin  $r = 0$  of the spherical coordinates. This is the so-called central shell-focusing singularity. Hence we will specify initial conditions only in a small neighborhood of the center and investigate the nature of the collapse in that region, without considering the evolution in the other regions of the spherical object. We will also assume that the initial conditions are such that shell-crossing singularities do not form during the evolution.

In Section 2 of the paper we write down the Einstein equations for the collapse of a bounded spherical object in an asymptotically flat spacetime. We then derive the equation for the acceleration of a fluid element in this spherical object, under the mutual influence of

gravity and the two pressures (radial and tangential). In Section 3 we use the acceleration equation to derive sufficient conditions for singularity formation, for the cases of collapse under tangential pressure, collapse under radial pressure and collapse of a perfect fluid. In all these cases we show that there are initial conditions for which the evolution is non-singular, and other initial conditions which result in singularity formation. In Section 4 we give an exact series solution for the evolution of the area radius, in the tangential pressure model under consideration. We show that the same solution also holds for a very specific kind of evolution under radial pressure or for a perfect fluid. In Section 5 we give a simplified derivation of the earlier results on naked singularity formation in the dust model. In Section 6 we show that covered as well as naked singularities form in the tangential pressure model as well as the radial pressure and perfect fluid models.

## 2 The Acceleration Equation

In comoving coordinates  $(t, r, \theta, \phi)$  the spherically symmetric line-element is given by

$$ds^2 = e^\sigma dt^2 - e^\omega dr^2 - R^2 d\Omega^2 \quad (1)$$

where  $\sigma$  and  $\omega$  are functions of  $t$  and  $r$ . The area radius  $R$  also depends on both  $t$  and  $r$ . In comoving coordinates the energy-momentum tensor for a spherically symmetric object takes the diagonal form  $T_k^i = (\rho, -p_r, -p_T, -p_T)$ . The quantities  $p_r$  and  $p_T$  are interpreted to be the radial and tangential pressure, respectively. The Einstein field equations for this system are

$$m' = 4\pi\rho R^2 R', \quad (2)$$

$$\dot{m} = -4\pi p_r R^2 \dot{R}, \quad (3)$$

$$\sigma' = -\frac{2p'_r}{\rho + p_r} + \frac{4R'}{R(\rho + p_r)}(p_T - p_r), \quad (4)$$

$$\dot{\omega} = -\frac{2\dot{\rho}}{\rho + p_r} - \frac{4\dot{R}(\rho + p_T)}{R(\rho + p_r)}, \quad (5)$$

and

$$m = \frac{1}{2}R \left( 1 + e^{-\sigma} \dot{R}^2 - e^{-\omega} R'^2 \right). \quad (6)$$

Here,  $m(t, r)$  is a free function arising out of integration of the Einstein equations. Its initial value,  $m(0, r)$ , is interpreted as the mass interior to the coordinate  $r$ .

In order to derive the equation for the acceleration we first define  $e^{-\omega} R'^2 = 1 + f(t, r)$ . Then

$$\frac{\dot{f}}{1+f} = -\dot{\omega} + \frac{2\dot{R}'}{R'} = \frac{\dot{R}}{R'} \sigma' \quad (7)$$

where the last equality follows from using (5) and eliminating  $\dot{\rho}$  using (2) and (3). Now we differentiate (6) w.r.t.  $t$  after writing it as

$$e^{-\sigma} \dot{R}^2 = \frac{2m}{R} + f \quad (8)$$

which gives the acceleration equation

$$\ddot{R} = -e^{\sigma} R \left( 4\pi p_r + \frac{m}{R^3} \right) + \frac{1}{2} \dot{\sigma} \dot{R} + \frac{e^{\sigma} (1+f)}{2R'} \left( -\frac{2p'_r}{\rho + p_r} + \frac{4R'(\rho + p_r)}{R(\rho + p_r)} \right). \quad (9)$$

By defining the proper time  $d\tau = e^{\sigma/2} dt$  this equation can also be written as

$$\frac{d^2 R}{d\tau^2} = -R \left( 4\pi p_r + \frac{m}{R^3} \right) + \frac{(1+f)}{2R'} \left( -\frac{2p'_r}{\rho + p_r} + \frac{4R'(p_T - p_r)}{R(\rho + p_r)} \right) \quad (10)$$

For a perfect fluid, we have  $p_T = p_r \equiv p$ , and the above equation becomes (see for instance [13]),

$$\frac{d^2 R}{d\tau^2} = -R \left( 4\pi p + \frac{m}{R^3} \right) - \frac{(1+f)}{R'} \frac{p'}{\rho + p}. \quad (11)$$

The Oppenheimer-Volkoff equation for hydrostatic equilibrium is obtained by setting the acceleration and the velocity equal to zero, and by noting, from (8), that  $f = -2m/R$ :

$$-R^2 \frac{dp}{dR} = m\rho \left[ 1 + \frac{p}{\rho} \right] \left[ 1 + \frac{4\pi R^3 p}{m} \right] \left[ 1 - \frac{2m}{R} \right]. \quad (12)$$

A few interesting properties about the role of pressure in the acceleration equation (9) should be noted. The tangential pressure appears only in the last term, and its gradient does not enter the equation. The gradient of only the radial pressure appears in the equation. A positive tangential pressure opposes collapse, while a negative tangential pressure supports it.

### 3 Conditions for singularity formation

There are various interesting special cases of the Einstein equations for spherical collapse given in the previous section, and we consider them one by one. The dust approximation is obtained by setting  $p_r = p_T = 0$ . In this case equation (2) remains as such, while equation (3) gives that the mass function does not depend on time  $t$ . Equation (4) implies that  $\sigma$  is a function only of time; hence we can redefine  $t$  and set  $\sigma = 0$ . Equation (5) can be integrated to get  $e^\omega = R'^2/(1+f(r))$ , where  $f(r)$  is a function of integration. Hence (6) can be written as

$$\dot{R}^2 = \frac{2m(r)}{R} + f(r). \quad (13)$$

The dust model has been discussed in detail by many authors [3].

### 3.1 Tangential Pressure

In the case when the radial pressure  $p_r$  is zero, and the tangential pressure non-zero, considerable simplification of the full system of equations takes place. As a result of equation (3), the mass function  $m(r)$  is time-independent. Equations (4) and (5) become

$$\sigma' = \frac{4R'}{R} \frac{p_T}{\rho}, \quad (14)$$

$$\dot{\omega} = -\frac{2\dot{\rho}}{\rho} - \frac{4\dot{R}}{R} \left(1 + \frac{p_T}{\rho}\right). \quad (15)$$

The equation (9) for acceleration becomes

$$\ddot{R} = -e^\sigma \frac{m}{R^2} + \frac{1}{2} \dot{\sigma} \dot{R} + \frac{2e^\sigma (1+f) p_T}{R} \frac{p_T}{\rho}. \quad (16)$$

We will assume that collapse begins from rest at time  $t = 0$ , and choose the initial scaling  $R(0, r) = r$ , which gives from equation (8) that initially  $f = -2m(r)/r$ . In order for collapse to begin, the acceleration must be negative, which implies that

$$\frac{2m}{R} > \frac{4p_T/\rho}{1 + 4p_T/\rho}. \quad (17)$$

Collapse will continue all the way up to the formation of a singularity  $R = 0$ , provided at any successive stage in the evolution, the acceleration is negative when  $\dot{R} = 0$ , i.e. provided (17) holds at all later times. The ratio  $p_T/\rho$  will in general evolve with time, for a given  $r$ . Let the initial value of this ratio be denoted by  $k(r)$ , assumed to be positive. A sufficient (though not necessary) condition for continual collapse is that the ratio  $p_T/\rho$  remains the same as, or falls below its initial value  $k(r)$ . If this happens, then (17) will be satisfied whenever  $\dot{R} = 0$ , because

$$\frac{2m(r)}{R} > \frac{2m(r)}{r} > \frac{4k(r)}{1 + 4k(r)} \geq \frac{4p_T/\rho}{1 + 4p_T/\rho}. \quad (18)$$

We now examine collapse with the assumption that the ratio  $p_T/\rho$  remains constant during evolution, at its initial value  $k(r)$ . This assumption makes the analysis tractable. The

constant  $k(r)$  is chosen to lie in the range  $0 < k(r) \leq 1$ . Since the tangential pressure must vanish at the origin as a result of isotropy, we must have  $k(0) = 0$ . Equation (5) has the solution

$$e^{-\omega(t,r)} = \chi(r)\rho^2 R^{4(1+k)}. \quad (19)$$

Here,  $\chi(r)$  is an arbitrary function of the coordinate  $r$ . Using equation (2) we can write this as

$$e^{-\omega(t,r)} = \frac{\chi(r)m'^2}{16\pi^2} \frac{R^{4k}}{R'^2} \equiv C(r) \frac{R^{4k}}{R'^2} = \frac{1+f(t,r)}{R'^2}. \quad (20)$$

Note that  $C(r)$  is a positive function. Using the solution (20) in equation (6) yields the following equation for the evolution of the area radius:

$$\left(\frac{dR}{d\tau}\right)^2 = \frac{2m}{R} - 1 + C(r)R^{4k}. \quad (21)$$

Assuming that collapse begins from rest gives

$$C(r) = \frac{\left(1 - \frac{2m}{r}\right)}{r^{4k}} \quad (22)$$

and hence (21) becomes

$$\left(\frac{dR}{d\tau}\right)^2 = \frac{2m}{R} - 1 + \left(1 - \frac{2m}{r}\right) \left(\frac{R}{r}\right)^{4k(r)}. \quad (23)$$

Let us consider the condition (17) for continual collapse. We are interested in the behaviour near the origin,  $r = 0$ . Near the origin, let  $k(r)$  behave as a power law,  $k(r) = ar^n$ . Since  $2m/r$  goes as  $r^2$  near the origin, collapse cannot take place if  $n = 1$ , and will necessarily take place if  $n \geq 3$ . The case  $n = 2$  is critical. Now, collapse will take place provided  $a < 2\pi\rho_0/3$ , but not otherwise. Here,  $\rho_0$  is the initial central density. Thus we find that for positive  $k$  there are singular as well as non-singular solutions. In Section 5 we will discuss conditions for the singularities to be covered or naked.



### 3.2 Perfect Fluid

The perfect fluid approximation is obtained by setting  $p_r = p_T = p$ . Equations (2), (3) and (6) remain as such, while equations (4) and (5) become

$$\sigma' = -\frac{2p'}{\rho + p}, \quad (24)$$

$$\dot{\omega} = -\frac{2\dot{\rho}}{\rho + p} - \frac{4\dot{R}}{R}. \quad (25)$$

The acceleration equation becomes

$$\frac{d^2 R}{d\tau^2} = -R \left( 4\pi p + \frac{m}{R^3} \right) - \frac{(1+f)}{R'} \frac{p'}{\rho + p}. \quad (26)$$

For collapse starting from rest at time  $t = 0$  and by using the initial scaling  $R = r$  we get that initially  $1 + f = 1 - 2m(r)/r$ . In order for the acceleration to be negative initially, we need

$$\frac{2m}{r} > \frac{1 + 4\pi p r (\rho + p)/p'}{1 - (\rho + p)/2r p'}. \quad (27)$$

Let the initial density profile near the center be

$$\rho = \rho_0 - \alpha r^n, \quad \alpha > 0, \quad n \geq 2 \quad (28)$$

and let the equation of state be  $p = k\rho$ , with  $k$  a positive constant. For  $n = 2$  the condition for negative initial acceleration is

$$\alpha < 2\pi(k+1)\rho_0(1 + \rho_0/3k). \quad (29)$$

For  $n \geq 3$  the initial acceleration is necessarily negative. It can be checked that a sufficient condition for singularity formation is that the quantity  $pR^2$  increases over, or remains the same as its initial value, and further, the quantity  $R|dp/dR|/(\rho + p)$  decreases from, or remains the same as, its initial value.

### 3.3 Radial Pressure

Corresponding equations may be written for the case when the tangential pressure  $p_T$  is zero, and collapse takes place only under radial pressure. It can be shown that starting from rest, the fluid will undergo collapse leading to singularity formation, provided the following condition is satisfied initially and subsequently:

$$\frac{2m}{R} > \frac{8\pi p_r R^2 + [2p'_r R + 4p_r R'] / (\rho + p_r) R'}{-1 + [2p'_r R + 4p_r R'] / (\rho + p_r) R'}. \quad (30)$$

Consider the initial density profile near the center to be of the form (28) (now with  $n \geq 1$ ), and an equation of state  $p_r = k(r)\rho$ . Since the radial pressure must vanish at  $r = 0$ ,  $k(r)$  must vanish at the origin. Let  $k(r) = Ar^n$  near the center. Then it follows from equation (30) that collapse will not take place for  $n = 1$ , will necessarily take place for  $n \geq 3$ , and if  $n = 2$ , collapse will take place provided  $\rho_0 > \pi/6A$ . A sufficient condition for singularity formation is that the quantity on the right hand side of (30) remains constant or falls below its initial value.

## 4 An exact solution

We will be interested in solving the collapse equation (23), which describes the collapse of a fluid starting from rest, subject to a tangential pressure equation of state  $p_T = k(r)\rho$ , with  $k(0) = 0$ . If this equation can be solved for a given  $m(r)$ , the remaining unknown functions ( $\rho, \omega, \sigma$  and  $p_T$ ) can also be obtained.

Interestingly enough, there is also a class of evolutions for the perfect fluid case and the radial pressure case, for which the equation for evolution of the area radius can be cast in a form similar to (23). Consider evolutions of the kind

$$8\pi p R^2 = \theta(r), \quad \sigma' = \psi(r) \frac{R'}{R}. \quad (31)$$

One cannot attach any physical importance to these assumptions, and they may at best only be approximately obeyed during the evolution. Their advantage is that one can then obtain a solution, subject to these assumptions, and demonstrate critical behaviour, and the occurrence of covered as well as naked singularities, in this solution. Our main purpose is to give a solution for the tangential pressure problem. It so happens that on the side, we can say something useful about a specific perfect fluid model, whose evolution equations are similar to the tangential pressure model. However, the fact that an exact solution is possible could be a motivation for understanding the assumptions on physical grounds. We would like to point out the following in that context.

It can be easily seen from equation (1) that  $\sigma$  plays the role of the Newtonian potential in the weak field limit, suggesting that  $\sigma'$  represents the gravitational pull on the source matter. One could imagine a fluid source for the Einstein field equations wherein the fluid particles were moving as shells of constant  $r$ . The mean curvature of these shells would then be  $R'/R$ . Also,  $8\pi pR^2$  is analogous to the force experienced by the body of fluid enclosed by the shell, solely due to the dynamics of the source particles.

With the assumptions made in (31) we can integrate equation (3) to get

$$2m(t, r) = \theta(r)(r - R) + 2m_0(r) \quad (32)$$

where  $m_0(r)$  is the initial mass distribution. Similarly, (7) and (31) can be used to get

$$1 + f(t, r) = A(r)R^{\psi(r)}, \quad (33)$$

where  $A(r)$  is an integration constant. Assuming that collapse begins from rest at  $t = 0$  (where the scaling is  $R = r$ ), we get that  $A(r) = (1 - 2m_0(r)/r)r^{-\psi}$ .

To rewrite equation (6), the following transformation is made use of:

$$d\tau = e^{\sigma/2}dt + Z(r)dr \quad (34)$$

where the requirement of exactness of the differential equation restricts the choice of  $Z(r)$  to

$$Z(r) = \int e^{\sigma/2} \frac{\sigma'}{2} + g(r) \quad (35)$$

$g$  being an arbitrary function of  $r$ . We aim at solving equation (6) keeping  $r$  fixed. We vary  $\tau$  under this restriction and examining equation (6) obtain

$$\left(\frac{dR}{d\tau}\right)^2 = \frac{2m_0}{R} - 1 + \theta(r) \left(\frac{r}{R} - 1\right) - \left(\frac{2m_0(r)}{r} - 1\right) \left(\frac{R}{r}\right)^{\psi(r)}. \quad (36)$$

In case of a perfect fluid, the assumptions in (31) reduce to

$$8\pi p R^2 = \theta(r), \quad -\frac{2R dp/dr}{R'(p+\rho)} = \psi(r). \quad (37)$$

For an equation of state  $p = k\rho$  and an initial density profile given by (28) near the center, we get that near  $r = 0$ ,

$$\psi(r) \sim \frac{n\alpha k}{(k+1)\rho_0} r^n. \quad (38)$$

After solving (36) for  $R(r, \tau)$  we can obtain the solutions for  $(\rho, \omega, \sigma$  and  $p_R)$ .

For collapse under radial pressure the assumptions (31) reduce to

$$8\pi p R^2 = \theta(r), \quad -\frac{4p'_r R + 8p_r R'}{R'(p+\rho)} = \psi(r). \quad (39)$$

For an equation of state of the form  $p_r = k(r)\rho$ , with  $k(r) = A_0 r^n$ , we get that  $\psi(r) = -4(n+2)k(r)$ .

The equation (23) for evolution of the area radius in the case of collapse with tangential pressure is a special case of (36), obtained by setting  $\theta(r) = 0$  and  $\psi(r) = 4k(r)$ . We now obtain an exact series solution of Equation (36), for a general  $\theta(r)$  and  $\psi(r)$ . After defining the variable  $y = R/r$  we write (36) as an integral:

$$\int d\tau = - \int \frac{r\sqrt{y}dy}{\sqrt{\theta + 2m_0/r}} \left(1 - \frac{1+\theta}{\theta + 2m_0/r} y + \frac{1 - 2m_0/r}{\theta + 2m_0/r} y^{1+\psi}\right)^{-1/2}. \quad (40)$$

By using the binomial expansion

$$(1-x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{1.3.5...(2n-1)}{2^n n!} x^n \quad (41)$$

and by taking

$$x = \frac{1+\theta}{\theta+2m_0/r} y + \frac{2m_0/r-1}{\theta+2m_0/r} y^{1+\psi} \quad (42)$$

the integral becomes

$$\int d\tau = -\frac{r}{\sqrt{\theta+2m_0/r}} \left[ \frac{2}{3} y^{3/2} + \sum_{n=1}^{\infty} \frac{1.3.5...(2n-1)}{2^n n!} \int x^n \sqrt{y} dy \right]. \quad (43)$$

We write  $x^n$  as

$$x^n = \frac{y^n}{(\theta+2m_0/r)^n} \sum_{j=0}^n {}^nC_j (1+\theta)^{n-j} \left( \frac{2m_0}{r} - 1 \right)^j y^{j\psi(r)} \quad (44)$$

and hence get the solution to the integral as

$$\tau - \tau_0(r) = -\frac{r}{\sqrt{\theta+2m_0/r}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{1}{(\theta+2m_0/r)^n} \times \quad (45)$$

$$\sum_{j=0}^n {}^nC_j (1+\theta)^{n-j} \left( \frac{2m_0}{r} - 1 \right)^j \frac{y^{n+j\psi+3/2}}{n+j\psi+3/2}. \quad (46)$$

This is an exact solution of the Einstein equation (36).

We cast this into the following form for convenience,

$$\tau - \tau_0(r) = -\frac{r}{\sqrt{\theta+2m_0/r}} y^{3/2} G(y, r) \quad (47)$$

where

$$G(y, r) = \sum_{n=0}^{\infty} \frac{(2n)}{2^{2n}(n!)^2} \frac{1}{(\theta+2m_0/r)^n} \times \quad (48)$$

$$\sum_{j=0}^n {}^nC_j (1+\theta)^{n-j} \left( \frac{2m_0}{r} - 1 \right)^j \frac{y^{n+j\psi+3/2}}{n+j\psi}.$$

It is useful to work out the derivatives of  $G$  :

$$\left(\frac{\partial G}{\partial y}\right)_r = -\frac{3}{2}\frac{G(y, r)}{y} + \frac{1}{y}\sqrt{\frac{2m_0/r + \theta - (1 + \theta)y - (2m_0 - 1)y^{\psi+1}}{2m_0/r + \theta}}, \quad (49)$$

$$\begin{aligned} \left(\frac{\partial G}{\partial r}\right)_y &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{1}{(\theta + 2m_0/r)^n} \times \\ &\sum_{j=0}^n {}^nC_j (1 + \theta)^{n-j} \left(\frac{2m_0}{r} - 1\right)^j \frac{y^{n+j\psi} D(y, r, n, j)}{n + j\psi + 3/2} \end{aligned} \quad (50)$$

where

$$\begin{aligned} D(y, r, n, j) &= (n - j) \frac{\theta'}{1 + \theta} + j \frac{2m'_0/r - 2m_0/r^2}{2m_0/r - 1} - n \frac{2m'_0/r - 2m_0/r^2 + \theta'}{2m_0/r + \theta} + \\ &j\psi' \log(y) - \frac{j\psi'}{n + j\psi + 3/2}. \end{aligned} \quad (51)$$

For the special case of dust, we have  $\theta = \psi = 0$ , and  $\tau = t$ . It can easily be shown from the dust equation (13) for the case of collapse starting from rest that the solution is given by

$$t - t_0(r) = -\frac{R^{3/2}}{\sqrt{2m(r)}} \left[ \frac{\arcsin \sqrt{y}}{y^{3/2}} - \frac{\sqrt{1-y}}{y} \right]. \quad (52)$$

The series solution (46) can be summed up in the dust case and it can be shown that the sum is equal to the closed form solution given in (52).

## 5 The Dust Solution

We will be interested in using the solution (46) to find out whether the singularity that forms at  $r = 0$  is covered or naked. We start by noting that in the cases of interest discussed above,  $\psi(r)$  has a power law form, near  $r = 0$ . Also,  $\theta(r)$  has a power law form near  $r = 0$ . Because of this, it can be inferred from (46) that as  $r \rightarrow 0$  the solution approaches the dust solution given in (52), with the difference that in (52),  $t$  should be replaced by  $\tau$  and  $2m(r)$  should be replaced by  $2m_0(r) + r\theta(r)$ , during the approach.

Hence it should be possible, as explicitly shown later, to use the naked singularity analysis for dust, carried out earlier, to draw conclusions about the occurrence of a naked singularity in collapse of a fluid under tangential pressure or radial pressure, or a perfect fluid, subject to the equation of state chosen above. Before we do so, we would like to use this opportunity to present a simplified derivation of the dust naked singularity. We give this simplified analysis first for the marginally bound case  $f(r) = 0$ , and then for the general case.

The solution to equation (13) in the marginally bound case is

$$R^{3/2} = r^{3/2} - \frac{3}{2}\sqrt{2m(r)}t. \quad (53)$$

An initial scaling  $R = r$  at the starting epoch  $t = 0$  of the collapse has been assumed. A curvature singularity forms at  $r = 0$  at time  $t_0 = 2/3\sqrt{\rho_0}$ , where  $\rho_0$  is the initial central density. We assume a series expansion near  $r = 0$  for the initial density function  $\rho(r)$ ,

$$\rho(r) = \rho_0 + \rho_1 r + \frac{1}{2!}\rho_2 r^2 + \frac{1}{3!}\rho_3 r^3 + \dots \quad (54)$$

The series expansion for the mass  $m(r)$  can then be deduced using (2).

From equation (53), we evaluate  $R'$  and then substitute for  $t$  from the same equation. In the resulting expression, substitute  $R = Xr^\alpha$ , and perform a Taylor expansion of  $F(r)$  around  $r = 0$ , so as to retain the leading non-vanishing term. We then get that near  $r = 0$ ,

$$\frac{R'}{r^{\alpha-1}} = X + \frac{\beta}{\sqrt{X}}r^{q+3/2-3\alpha/2}. \quad (55)$$

Here,  $\beta = -2qm_q/3m_0$ , and  $q$  is defined such that in a series expansion of the initial density  $\rho(r)$  near the center, the first non-vanishing derivative is the  $q$ th one ( $=\rho_q$ ), and  $m_q = 4\pi\rho_q/(q+3)q!$ . It may be shown that  $\alpha$  has a unique value at the approach to the central singularity, given by setting the power of  $r$  to zero in the second term, i.e.  $\alpha = 1 + 2q/3$ . This reproduces the result of the  $R'$  calculation performed earlier [14], in a simpler manner.

For the non-marginally bound case (i.e.  $f \neq 0$ ) the solution of the Tolman-Bondi equation is

$$R^{3/2}G(-fR/F) = r^{3/2}G(-fr/F) - \sqrt{F(r)}t \quad (56)$$

where  $F(r) = 2m(r)$ , and  $G(y)$  is a positive function having the domain  $1 \geq y \geq -\infty$  and is given by.  $R'$  can be evaluated as before, and then we eliminate  $t$  and substitute  $R = Xr^\alpha$ . The power-series expansion for  $f(r)$  near  $r = 0$  is of the form [14]

$$f(r) = f_2r^2 + f_3r^3 + f_4r^4 + \dots \quad (57)$$

which implies that the argument  $(-fR/F)$  of  $G$  on the left-side of (56) goes to zero as  $r \rightarrow 0$ . The derivative of  $G(-p)$  w.r.t. its argument  $p \equiv rf/F$  is obtained by differentiating (56), which gives

$$\frac{dG(-p)}{d(-p)} = \frac{3G}{2p} - \frac{1}{p\sqrt{1+p}}.$$

Using this, one can now perform a Taylor-expansion of  $G, F$  and  $f$  about  $r = 0$  to get exactly the same expression for  $R'$  as given in (55) above, where  $\beta$  is now given by

$$\beta = q \left( 1 - \frac{f_2}{2F_0} \right) \left( G(-f_2/F_0) \left[ \frac{F_q}{F_0} - \frac{3f_{q+2}}{2f_2} \right] \left[ 1 + \frac{f_2}{2F_0} \right] + \frac{f_{q+2}}{f_2} - \frac{F_q}{F_0} \right). \quad (58)$$

The constant  $q$  is now defined such that the first non-vanishing derivative of the initial density is the  $q$ th one, and/or the first non-vanishing term in the expansion for  $f(r)$ , beyond the quadratic term, is of order  $r^{q+2}$ . The constant  $\alpha$  is again equal to  $(1 + 2q/3)$ . Thus the  $R'$  calculation is simplified for the non-marginal case as well.

In order for the singularity at  $r = 0$  to be naked, radial null geodesics should be able to propagate outwards, starting from the singularity. A necessary and sufficient condition



for this to happen is that the area radius  $R$  increase along an outgoing geodesic, because  $R$  becomes negative in the unphysical region. Thus we write, along the geodesic,

$$\frac{dR}{dr} = R' + \dot{R} \frac{dt}{dr} = R' \left( 1 + \frac{\dot{R}}{\sqrt{1+f}} \right) = R' \left( 1 - \sqrt{\frac{f + F/R}{1+f}} \right). \quad (59)$$

Here we have substituted  $dt/dr = R'/\sqrt{1+f}$  along an outgoing null ray (using the metric (1)) and substituted for  $\dot{R}$  from (13).  $dR/dr$  should be positive along the outgoing geodesic. We now define  $u = r^\alpha$ , and use  $X$  as a variable, instead of  $R$ . Hence, in the approach to the singularity, (i.e. as  $R \rightarrow 0, r \rightarrow 0$ ),  $X$  takes the limiting value  $X_0$  given by

$$X_0 = \lim_{r \rightarrow 0, R \rightarrow 0} \frac{R}{u} = \lim_{r \rightarrow 0, R \rightarrow 0} \frac{1}{\alpha r^{\alpha-1}} \frac{dR}{dr} = \lim_{r \rightarrow 0, R \rightarrow 0} \frac{1}{\alpha r^{\alpha-1}} R' \left( 1 - \sqrt{\frac{f + F/R}{1+f}} \right). \quad (60)$$

By using (55) we can write

$$X_0 = \frac{1}{\alpha} \left( X_0 + \frac{\beta}{\sqrt{X_0}} \right) \left( 1 - \sqrt{\frac{f(0) + \Lambda_0/X_0}{1+f(0)}} \right) \quad (61)$$

The constant  $\Lambda_0$  is the limiting value of  $\Lambda(r) = F(r)/r^\alpha$  as  $r \rightarrow 0$ . The variable  $X$  can be interpreted as the tangent to the outgoing geodesic, in the  $R, u$  plane. As can be seen from the above, the positivity of  $dR/dr$  along an outgoing geodesic is equivalent to requiring that the equation (61) admit a positive root  $X_0$ . This will depend on the initial density and velocity distribution, which determine the functions  $F(r)$  and  $f(r)$ , and hence the functions  $\beta$  and  $\Lambda$ . One can solve equation (61), and as shown in [14], the results for the nature of the singularity are the following:

In the marginally bound case, the singularity is naked if  $\rho_1 < 0$ , or if  $\rho_1 = 0, \rho_2 < 0$ . If  $\rho_1 = 0$  and  $\rho_2 = 0$ , one defines the quantity  $\zeta = 2m_3/(2m_0)^{5/2}$ . The singularity is naked if  $\zeta \leq -25.9904$  and covered if  $\zeta$  exceeds this value. If  $\rho_1 = \rho_2 = \rho_3 = 0$ , the singularity is covered. In the non-marginally bound case, if  $F_1$  and  $f_3$  are non-zero, the singularity is naked if

$$Q_1 = \left( 1 - \frac{f_2}{2F_0} \right) \left( G(-f_2/F_0) \left[ \frac{F_1}{F_0} - \frac{3f_3}{2f_2} \right] \left[ 1 + \frac{f_2}{2F_0} \right] + \frac{f_3}{f_2} - \frac{F_1}{F_0} \right) \quad (62)$$

is positive. If  $F_1$  and  $f_3$  are both zero, and  $F_2$  and  $f_4$ , are non-zero the singularity is naked if

$$Q_2 = \left(1 - \frac{f_2}{2F_0}\right) \left(G(-f_2/F_0) \left[\frac{F_2}{F_0} - \frac{3f_4}{2f_2}\right] \left[1 + \frac{f_2}{2F_0}\right] + \frac{f_4}{f_2} - \frac{F_2}{F_0}\right) \quad (63)$$

is positive. If  $F_1$ ,  $f_3$ ,  $F_2$  and  $f_4$  are zero and if  $F_3$  and  $f_5$  are non-zero, the singularity is naked if

$$Q_3 = \left(1 - \frac{f_2}{2F_0}\right) \left(G(-f_2/F_0) \left[\frac{F_3}{F_0} - \frac{3f_5}{2f_2}\right] \left[1 + \frac{f_2}{2F_0}\right] + \frac{f_5}{f_2} - \frac{F_3}{F_0}\right) \quad (64)$$

is positive. If  $F_1$ ,  $f_3$ ,  $F_2$ ,  $f_4$ ,  $F_3$  and  $f_5$  are all zero, the singularity is covered.

A special case of non-marginally bound collapse is the collapse starting from rest, for which  $f(r) = -2m(r)/r$ . As a result,  $G(-fR/F) = \pi/2$  and the calculation of  $R'$  in (56) can be carried out exactly as in the marginally bound case, to get the result (55), with  $\beta = -\pi q m_q/4m_0$ . The conditions for a naked singularity to occur are the same as those stated above for marginally bound collapse, except that  $\zeta$  is defined as  $\zeta = 6\pi m_3/4(2m_0)^{5/2}$ .

We also point out that in a recent work [15] we have given a yet simpler derivation of the dust naked singularity, which directly looks for a self-consistent solution of the geodesic equation, in a neighborhood of the singularity. In principle, the method described in [15] can be applied also to the tangential pressure solution discussed here.

## 6 The occurrence of covered and naked singularities

We can now utilize the dust results for inferring the occurrence of covered and naked singularities in collapse with the equations of state considered here, except that two further subtleties remain to be sorted out. Firstly, the solution (46) is written with respect to the time variable  $\tau$ , for a *fixed*  $r$ . We need to show that a relation similar to equation (60) holds. Secondly, for a general  $\psi$  and  $\theta$  (i.e. not restricted to power law forms for instance) the dependence of  $f$  and  $F$  on  $t$  as well introduces terms in  $X_0$  different from the dust case. It is not obvious a priori that these vanish as one approaches the singularity, even though the

solution tends to behave like dust in this limit. It needs to be explicitly shown that a relationship identical in form to equation (61) holds in the general case. This is demonstrated in the rest of the section.

By requiring that  $d\tau$  be integrable, we can write

$$d\tau = e^{\sigma/2} dt + Z(r) dr = e^{\sigma/2} dt + \left[ \int e^{\sigma/2} \frac{\sigma'}{2} dt + g(r) \right] dr \quad (65)$$

where the last expression follows from using the integrability condition on  $Z(r)$ , and  $g(r)$  is an arbitrary function. We choose  $g(r)$  in such a way that in the approach to the central singularity the coefficient of  $dr$  in (65) vanishes. As a result, in the approach to the singularity we can write  $d\tau = e^{\sigma/2} dt$ .

The following can be easily shown from (36) and the solution (47).

$$\left( \frac{\partial R}{\partial \tau} \right)_r = -\frac{1}{\sqrt{y}} [A(y, r)]^{1/2} \quad (66)$$

where

$$A(y, r) = \frac{2m_0(r)}{r} + \theta - (1 + \theta)y - \left( \frac{2m_0}{r} - 1 \right) y^{1+\psi}. \quad (67)$$

$\partial R / \partial \tau$  is finite in the approach to the singularity. Also,

$$\left( \frac{\partial R}{\partial r} \right)_\tau = y \left[ 1 + \frac{r}{\sqrt{2m_0/r + \theta}} \sqrt{A(y, r)} B(y, r) \right] \quad (68)$$

where

$$B(y, r) = -\frac{\tau'_0 G}{\tau - \tau_0} - \frac{G}{r} + 1/2 \frac{2m'_0/r - 2m_0/r^2 + \theta'}{2m_0/r + \theta} - \left( \frac{\partial G}{\partial r} \right)_y. \quad (69)$$

Now,

$$(R')_t = (R')_\tau + \left( \frac{\partial R}{\partial \tau} \right)_r \left( \frac{\partial \tau}{\partial r} \right)_t \quad (70)$$

and it follows from the above discussion that the last term,  $M(y, r)$  say, can be made to vanish in the approach to the singularity, so that in the limit the quantities  $(R')_t$  and  $(R')_\tau$  become identical to each other.

From assumptions (31) one can easily show that

$$R'^2 e^{-\omega} = (1 - 2m_0/r) y^\psi \quad (71)$$

This leads to

$$e^{\omega/2} = \frac{y^{1-\psi/2}}{\sqrt{1-2m_0/r}} \left[ 1 + \frac{r}{\sqrt{2m_0/r+\theta}} \sqrt{A(y,r)B(y,r)} \right] + \frac{y^{1-\psi/2}}{\sqrt{1-2m_0/r}} M(y,r) \quad (72)$$

From (65) we also get that along a null geodesic

$$\frac{\partial \tau}{\partial r} = e^{\omega/2} + \int e^{\sigma/2} \frac{\sigma'}{2} dt + g(r). \quad (73)$$

This, after using (72), is re-written as

$$\begin{aligned} \frac{\partial \tau}{\partial r} = & \frac{y^{1-\psi/2}}{\sqrt{1-2m_0/r}} \left[ 1 + \frac{r}{\sqrt{2m_0/r+\theta}} \sqrt{A(y,r)B(y,r)} \right] + \frac{y^{1-\psi/2}}{\sqrt{1-2m_0/r}} M(y,r) \\ & + \int e^{\sigma/2} \frac{\sigma'}{2} dt + g(r) \end{aligned} \quad (74)$$

Hence we may examine the rate of change of  $R$  along an outgoing null ray as

$$\left( \frac{dR}{dr} \right)_{\text{null geodesic}} = (R')_\tau + \left( \frac{\partial R}{\partial \tau} \right)_{\text{null geodesic}} \left( \frac{\partial \tau}{\partial r} \right) \quad (75)$$

From equations (66), (68) and (74), this can be cast as

$$\begin{aligned} \left( \frac{dR}{du} \right)_{\text{null geodesic}} = & \frac{1}{\alpha} \left[ X - \sqrt{\frac{A}{(2m_0/r+\theta)r^{3(\alpha-1)}}} \frac{\tau'_0}{\sqrt{X}} - \sqrt{\frac{A}{2m_0/r+\theta}} \sum_{n=0}^{\infty} \frac{(2n)}{2^{2n}(n)^2} \frac{1}{(\theta+2m_0/r)^n} \right. \\ & \left. \sum_{j=0}^n X^{n+j\psi} {}^n C_j (1+\theta)^{n-j} \left( \frac{2m_0}{r} - 1 \right)^j \frac{r^{(\alpha-1)(n+j\psi)}}{n+j\psi+3/2} \right. \\ & \left. \left( D(y,r,n,j) + \frac{1}{r} + 1/2 \frac{2m'_0/r - 2m_0/r^2 + \theta'}{2m_0/r+\theta} \right) \right] \left( 1 - X^{(\psi+1)/2} \sqrt{\frac{A}{(2m_0/r-1)(r^{(\alpha-1)(n+j\psi)})}} \right) \end{aligned} \quad (76)$$

where the derivative on the left is to be evaluated along null geodesics. This resembles the roots equation (59).

Indeed, the first term with square parenthesis in equation (76) is similar to  $R'/r^{\alpha-1}$  in equation (60). It turns out that as one takes the limit  $r \rightarrow 0$ , the third contribution within this term containing the summation vanishes for physically reasonable choices of  $\psi$ , i.e. vanishes as one takes the limit. This requirement is certainly satisfied by the three special cases in Sections 3.1, 3.2 and 3.3. In the rest of the contribution, these conditions imply that the result is the same as dust, save the replacement of  $2m$  by  $2m_0 + r\theta$  and  $t_0$  by  $\tau_0$  in the approach to the singularity. Similarly, the second term in square parenthesis in equation (76) has a counterpart in the dust case, although the familiar  $f$  and  $F$  of (59) have no individual correspondences. However, if one examines the limiting form (60) of dust and works out (76) in the limit, then one finds that  $f(0)$  of (60) corresponds to  $-2m_0/r - \theta$  evaluated in the limit  $r = 0$  and  $\Lambda_0$  gets replaced by  $(2m_0 + r\theta)/r^\alpha$  (evaluated in the limit  $r = 0$ ). Hence we arrive at the same roots equation as (61).

It is important to realize that the first subtlety mentioned at the beginning is sorted out by a judicious choice of the  $\tau$  coordinate and results in the fact that  $(\partial R/\partial r)_\tau$  tends to  $R'$  of the dust case in (59) in the limit. The second subtlety presents itself explicitly when it turns out that the correspondences of  $f$  and  $F$  of dust are not  $f(r, t)$  and  $F(r, t)$  of the general case if one has not taken the limit. It would be therefore in general incorrect to conclude equation (76) to be equation (59) with  $f$  and  $F$  simply replaced by their generalizations, at this stage. However, the term  $(f + F/R)/(1 + f)$  in (59) still corresponds to the quantity obtained by simple generalization of the free functions of dust. The existence of such a quantity is not apriori guaranteed when one notices that the solution (46) behaves similar to the dust solution in the approach to the singularity.

We now discuss the nature of the singularities. Consider first collapse under tangential pressure  $p_T$  with the equation of state  $p_T = k(r)\rho$ , which we have discussed above. Since in

this case  $\theta(r) = 0$  it follows from the exact solution given above in (46) that if the collapse ends in a singularity, then the conditions for the occurrence of covered and naked singularities are exactly the same as in the case of dust collapse starting from rest. The introduction of a tangential pressure (which must vanish at the center even though it is non-zero elsewhere) does not change the nature of the dust singularity. Further, we expect that even if we do not restrict to the case of linear equation of state, the conditions for a singularity to be naked in the tangential pressure model will be exactly the same as in the dust case, so long as the tangential pressure vanishes at the origin.

The situation is more interesting in the perfect fluid case, because now  $\theta(r) \neq 0$ . The leading order solution is the dust solution, but with an “effective mass”  $2m_0(r) + \theta(r)$ . Hence the dust results on the nature of the singularity are applicable, except that we must replace  $2m_0(r)$  by  $2m_0(r) + \theta(r)$ . Recalling the definition of  $\theta(r)$ , we can expand it in a series near  $r = 0$ ,

$$\frac{\theta(r)}{4\pi} = p_0 r^2 + \frac{1}{2} p_2 r^2 + \frac{1}{6} p_3 r^6 + \dots \quad (77)$$

We have also assumed an equation of state  $p = k\rho$ , with  $k$  a constant. Hence the coefficients in the above expansion for the pressure are related to those for the density in the expansion in (54). Eqns. (28) and (29) give the conditions for singularity formation, if  $\rho_2 < 0$ . If a singularity does form it will be naked. If  $\rho_2 = 0$  and  $\rho_3 < 0$  a singularity will necessarily form. It will be naked if  $\zeta = 3\pi(2m_3 + 2\pi p_3/3)/4F_0^{5/2}$  is less than  $-25.9904$  and covered if  $\zeta$  exceeds this value. Thus we find that in the case of a perfect fluid, the condition for the occurrence of a naked singularity differs from that in the dust case, because of the presence of the constant  $k$  in the definition for  $\zeta$ . By following a similar series of arguments one can conclude that if a singularity forms in the radial pressure model considered above, the conditions for it to be naked or covered are exactly the same as in the dust case.

## 7 Appendix

Although we have shown that the collapse of the cloud in Sections 3.1, 3.2 and 3.3 leads to the formation of a singularity for some initial conditions, it is necessary to ensure that the same happens in case of the general solution (46) obtained with the assumptions (31). We proceed in a manner similar to these cases. Initially, let  $\dot{R}$  be 0 and  $\ddot{R}$  be negative. If the collapse has to take place without any rebounds hereafter, one needs to ensure that the shells have negative ‘acceleration’  $\ddot{R}$  whenever they reach the velocity limit  $\dot{R} = 0$ . This along with the initial condition keeps  $\dot{R}$  non-positive. It is straightforward to show from assumptions (31) using (71) and (6) that this requirement for avoiding rebounds reduces to the following condition on  $R$ :

$$\frac{R}{r} < \frac{\frac{\psi}{2} (2m_0/r + \theta)}{(2m_0/r + \theta) + (1 + \theta) \frac{\psi}{2}}$$

whenever  $R$  satisfies

$$2m_0/r + \theta - (1 + \theta) \frac{R}{r} + (1 - 2m_0/r) \left( \frac{R}{r} \right)^{1+\psi} = 0.$$

Note that this is equivalent to an algebraic inequality dependent solely on the initial data, when the root  $R/r$  in the equation above is bounded by the first inequality. Let us suppose this constraint on the initial free functions is satisfied for one of the roots which we know to be 1. This is nothing but the initial epoch of the collapse. The collapse begins. Let us suppose there occurs another real root (i.e.  $\dot{R}$  vanishes again). This time the collapse has proceeded and the root, therefore, has to be lesser than 1. The first inequality, already satisfied for  $R/r = 1$  will therefore be automatically satisfied making the acceleration negative in this situation. Thus the rebound will be prevented. Hence we conclude that if we ensure that the initial data is chosen such that the cloud begins to collapse at the first instant, then it is implied that the cloud will have no rebounds at all at any later instant.

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